Explicit solutions of the classical Calogero & Sutherland systems for any root system

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Abstract

Explicit solutions of the classical Calogero (rational with/without harmonic confining potential) and Sutherland (trigonometric potential) systems is obtained by diagonalisation of certain matrices of simple time evolution. The method works for Calogero & Sutherland systems based on any root system. It generalises the well-known results by Olshanetsky and Perelomov for the A type root systems. Explicit solutions of the (rational and trigonometric) higher Hamiltonian flows of the integrable hierarchy can be readily obtained in a similar way for those based on the classical root systems.

1 Introduction

The classical and quantum integrability/solvability of Calogero-Moser systems [1]–[5] manifests itself in many guises; the existence of Lax pairs and/or Dunkl operators, algebraic linearisation, quadratic algebras, associated integrable spin chains, 'quantised' classical spectra, etc. Among them, a most intuitive understanding of solvability/integrability is provided by the fact that explicit solutions of the classical equations of motion are obtained by diagonalisation of certain matrices having trivial time evolution as shown by Olshanetsky and Perelomov [6] for the rational and trigonometric potential cases [7]. Their results are for the systems based on the A type roots. Here we will show that the same results hold universally for systems based on any root system. To be more precise, for the rational potential (without/with harmonic confining potential) cases, the diagonalisation method works for any root

system including the non-crystallographic ones. For the trigonometric and hyperbolic potential cases, we show that the explicit diagonalisation method holds for any crystallographic root system based on the *universal* Lax pair [8]. A simpler form of explicit diagonalisation is provided by the *minimal* Lax pair [9, 8], which exists only for those based on the A, D, E_6 and E_7 root systems. The basic idea of the explicit solution method is very closely related to the notion of algebraic linearisation proved universally for any root system by Caseiro-Françiose-Sasaki [10]. We will follow the notation of [10] throughout this paper and eq.(a.b) of this paper will be cites as (Ia.b). Explicit solutions in terms of diagonalisation is readily obtained for the *higher* (rational and trigonometric) *Hamiltonian flows* belonging to the integrable hierarchy. This works, however, only for those based on the classical root systems, A, B, C and D. The conventional Lax pair in terms of the set of vector weights (A, C and D) or the set of short roots (B) is indispensable.

This paper is organised as follows. In section two, the historical background and the logical structure of the Calogero-Moser systems necessary for the present paper are briefly reviewed. The Hamiltonian and the universal Lax pair with rational potential are introduced. The explicit integration in terms of diagonalisation is achieved by relating the Lax pair matrices L and M to a matrix W of the same size with trivial time evolution, $\ddot{W} = 0$. Section three is devoted to the explicit solution of the systems with rational plus the harmonic confining potential. In section four, we show the explicit solutions of the Sutherland systems, which have trigonometric/hyperbolic potentials. In §4.1 a simple form of explicit diagonalisation is obtained by re-interpreting the formulas of the algebraic linearisation method developed by Caseiro-Françoise-Sasaki [10]. This is based on the minimal Lax pair, which exists for those based on A, D, E_6 and E_7 root systems. A general treatment of explicit integration of the Sutherland systems in terms of the universal Lax pair is provided in §4.2. This applies to any crystallographic root system. Sections five and six are devoted to the problem of explicit integration of the higher Hamiltonian flows of the integrable hierarchy. The rational potential case is discussed in section five and the trigonometric case in section six. The final section is for summary and comments.

2   Rational Potential

The integrability of the Calogero-Moser systems has a long history. Firstly, various types of the integrable potentials are recognised; starting from the Calogero model [1] with rational

$(1/q^2)$ plus a harmonic confining (q^2) potential followed by the Sutherland model [2] with a trigonometric $(1/\sin^2 q)$ potential. Then the pure rational potential $(1/q^2)$ [3] and the hyperbolic $(1/\sinh^2 q)$ [4] and the elliptic $(\wp(q))$ potentials [5] are added to the list of the integrable potentials. As seen in many other subjects in mathematical physics, the quantum groups, the integrable spin chains, Yang-Baxter equations, etc., the elliptic case is the generic one giving all the rest in various degenerations. However, each degenerate case, the rational, trigonometric and the hyperbolic, has its own special properties and merits not shared by the more generic ones; for example, the algebraic linearisation [10] of the degenerate Calogero-Moser systems and the quadratic algebras [11] for the quantum systems with the super-integrable rational $(1/q^2)$ potential. The present article deals with one of such properties of the degenerate Calogero-Moser systems and it is in fact very closely related to the algebraic linearisation [10]. Secondly, the nature of the multi-particle interactions of the Calogero-Moser systems is recognised to be governed by the root systems associated with finite reflection (Coxeter/Weyl) groups [12, 13]. The original models [1]–[5] are all based on the A type root system related to the symmetric group S_N with N being the number of the particles. The S_N is also the Weyl group of the special unitary group $SU(N)$. The integrability (the Lax pair) of the systems based on the classical root systems (A, B, C and D) is noticed immediately by Olshanetsky and Perelomov [12, 13], but the actual demonstration of the integrability of the Calogero-Moser systems based on the exceptional [9, 14] and non-crystallographic root systems [8] took longer years. The classical universal Lax pair applicable for all types of potentials and for any root system [8] and the quantum universal Lax pair applicable for all degenerate potentials and for any root system [15] have been known for some years.

Let us denote by Δ a root system of rank r. It is a finite set of $\mathbf{R}^r$ vectors which is invariant under reflections in the hyperplane perpendicular to each vector in Δ. A reflection s_ρ in terms of a root ρ is defined by

$$s_\rho(x) = x - \rho\,(\rho^\vee \cdot x), \quad x \in \mathbf{R}^r, \tag{2.1}$$

in which $\rho^\vee = 2\rho/\rho^2$. Thus Δ is characterised by

$$s_\rho(\eta) \in \Delta, \quad \forall \rho, \eta \in \Delta. \tag{2.2}$$

The dynamical variables are the coordinates $q_i \in \mathbf{R}$, $i = 1, ..., r$ and their canonically conjugate momenta $p_i \in \mathbf{R}$, $i = 1, ..., r$, except for the A_r case in the ordinary embedding,

in which the number of particles is $r + 1$. The Hamiltonian for the classical Calogero-Moser system with the rational potential but without the harmonic confining potential is:

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2}\sum_{\rho \in \Delta_+} \frac{g_{|\rho|}^2 |\rho|^2}{(\rho \cdot q)^2},\tag{2.3}$$

in which the real and positive coupling constants $g_{|\rho|}$ are defined on orbits of the corresponding Coxeter group. That is, for the simple Lie algebra cases $g_{|\rho|} = g$ for all roots in simply-laced models and $g_{|\rho|} = g_L$ for long roots and $g_{|\rho|} = g_S$ for short roots in non-simply laced models. In order to define Lax pair matrices L and M, let us choose a set of $\mathbf{R}^r$ vectors $\mathcal{R} = \{\alpha, \beta, \ldots, \}$, $\#\mathcal{R} = \mathcal{D}$, permuting under the action of the reflection group:

$$s_\rho(\alpha) \in \mathcal{R}, \quad \forall \alpha \in \mathcal{R}, \quad \forall \rho \in \Delta.\tag{2.4}$$

We demand that it consists of a single orbit of the Coxeter group, for irreducibility. Then we define $\mathcal{D} \times \mathcal{D}$ matrices indexed by the elements of $\mathcal{R}$:

$$p \cdot \hat{H}: \quad (p \cdot \hat{H})_{\alpha\beta} = (p \cdot \alpha)\,\delta_{\alpha\beta},\tag{2.5}$$

$$\hat{s}_\rho: \quad (\hat{s}_\rho)_{\alpha\beta} = \delta_{\alpha,s_\rho(\beta)}.\tag{2.6}$$

Introduce next the $\mathcal{D} \times \mathcal{D}$ matrices X, L and M [9, 8]:

$$X = i\sum_{\rho \in \Delta_+} g_{|\rho|}(\rho \cdot \hat{H})\frac{1}{(\rho \cdot q)}\,\hat{s}_\rho,\tag{2.7}$$

$$L = p \cdot \hat{H} + X,\tag{2.8}$$

$$M = -\frac{i}{2}\sum_{\rho \in \Delta_+} g_{|\rho|}\frac{|\rho|^2}{(\rho \cdot q)^2}\,\hat{s}_\rho,\tag{2.9}$$

and a *diagonal matrix*:

$$Q = q \cdot \hat{H}: \quad (Q)_{\alpha\beta} = (q \cdot \alpha)\,\delta_{\alpha\beta}.\tag{2.10}$$

Here L and Q are hermitian $L^\dagger = L$, $Q^\dagger = Q$ and M is anti-hermitian $M^\dagger = -M$.

As shown in [10] (I2.7a), (I2.7b) the time evolution of the matrix L along the flow of the Hamiltonian (2.3) displays the following equations:

$$\frac{\partial L}{\partial t} = [L, M],\tag{2.11}$$

$$\frac{\partial Q}{\partial t} = [Q, M] + L.\tag{2.12}$$

Next let us define another $\mathcal{D} \times \mathcal{D}$ unitary matrix $U(t)$ by the linear equation and the initial condition:

$$\frac{\partial U}{\partial t} = UM, \qquad U(0) = 1_{\mathcal{D}}, \tag{2.13}$$

in which $1_{\mathcal{D}}$ is the $\mathcal{D} \times \mathcal{D}$ unit matrix. The final step is the introduction of $W(t)$:

$$W(t) \equiv U(t)Q(t)U^{-1}(t), \tag{2.14}$$

which has a simple time evolution

$$\dot{W} = U(\dot{Q} - [Q, M])U^{-1} = ULU^{-1}, \tag{2.15}$$

$$\ddot{W} = U(\dot{L} - [L, M])U^{-1} = 0. \tag{2.16}$$

The solution is

$$W(t) = W(0) + t\,\dot{W}(0), \tag{2.17}$$

with the initial values

$$W(0) = Q(0), \quad \dot{W}(0) = L(0), \tag{2.18}$$

which are determined by the initial values of the canonical variables $q_j(0)$, $p_j(0)$, $j = 1, \ldots, r$. Due to the defining relation of $W(t)$ in terms of the *diagonal matrix* $Q(t)$ (2.14), the solution $\{q(t)\}$ of the the canonical equations of motion

$$\frac{\partial q_j}{\partial t} = \frac{\partial \mathcal{H}}{\partial p_j}, \qquad \frac{\partial p_j}{\partial t} = -\frac{\partial \mathcal{H}}{\partial q_j}, \qquad j = 1, \ldots, r, \tag{2.19}$$

with the above Hamiltonian (2.3), is simply obtained by *diagonalising* the above matrix solution (2.17). The conjugate momenta $\{p(t)\}$ are obtained by differentiation $p_j(t) = \partial q_j(t)/\partial t$.

As promised, this is the universal proof applicable for any root system including the non-crystallographic one. The spectrum of $W(t)$ (2.17) is highly constrained, since its dimension $\mathcal{D}$ is usually much greater than the degree of freedom r. The high symmetry of the spectrum is guaranteed by the Coxeter invariance of the theory:

$$\mathcal{H}(s_\rho(p), s_\rho(q)) = \mathcal{H}(p, q), \qquad \forall \rho \in \Delta, \tag{2.20}$$

$$L(s_\rho(p), s_\rho(q)) = \hat{s}_\rho L(p, q)\hat{s}_\rho, \quad M(s_\rho(q)) = \hat{s}_\rho M(q)\hat{s}_\rho. \tag{2.21}$$

The original proof of the explicit integration of the A type systems by Olshanetsky and Perelomov [6] is the very special case in which the spectrum of $W(t)$ (2.17) is not constrained. Our proof reduces to that of [6] when $\Delta = A_r$ and the set of vector weights is chosen as $\mathcal{R} = \mathbf{V}$, $\#\mathcal{R} = \mathcal{D} = r + 1$.

3 Rational with Harmonic Confining Potential

The Hamiltonian is now:

$$\mathcal{H}_\omega = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{1}{2}\sum_{\rho \in \Delta_+} \frac{g_{|\rho|}^2 |\rho|^2}{(\rho \cdot q)^2}. \tag{3.1}$$

With the same matrices introduced above in the preceding section, the time evolution displays (I3.2a), (I3.2b):

$$\dot{L} = [L, M] - \omega^2 Q, \tag{3.2}$$

$$\dot{Q} = [Q, M] + L. \tag{3.3}$$

With the same definition of the unitary matrix $U(t)$ as above (2.13), the matrix

$$W(t) \equiv U(t)Q(t)U^{-1}(t), \tag{3.4}$$

evolves harmonically in time:

$$\dot{W} = U(\dot{Q} - [Q, M])U^{-1} = ULU^{-1} \tag{3.5}$$

$$\ddot{W} = U(\dot{L} - [L, M])U^{-1} = -\omega^2 W. \tag{3.6}$$

The solution is

$$W(t) = \cos \omega t\, W(0) + \omega^{-1} \sin \omega t\, \dot{W}(0), \tag{3.7}$$

with the initial values

$$W(0) = Q(0), \quad \dot{W}(0) = L(0). \tag{3.8}$$

Again the explicit solution $\{q(t)\}$ is obtained by *diagonalising* the above matrix $W(t)$ (3.7) with the harmonic time dependence.

4 Trigonometric Potential

The Hamiltonian of the trigonometric (Sutherland) model [2] writes:

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2}\sum_{\rho \in \Delta_+} \frac{g_{|\rho|}^2 |\rho|^2}{\sin^2(\rho \cdot q)}. \tag{4.1}$$

In order to get the hyperbolic case it suffices to change sin into sinh. In the following, we only demonstrate the explicit integration of the trigonometric case. The hyperbolic case can be deduced easily by the above replacement.

Two types of Lax pairs are known [9, 8] for the trigonometric cases: the minimal and the universal Lax pairs. While the latter, the universal lax pair, applies to any crystallographic root system, the former, the minimal Lax pair, requires $\mathcal{R}$ to be the set of minimal weights, which exists only for the A, D, E_6 and E_7 root systems. Let us start with the minimal Lax pair which has simpler structure thanks to the restriction to the minimal weights, satisfying the condition:

$$\mu : \text{ minimal weight } \Leftrightarrow \alpha^\vee \cdot \mu = 0, \pm 1, \quad \forall \alpha \in \Delta. \tag{4.2}$$

4.1 Minimal Lax pair

We consider the matrices [9, 8]:

$$L = p \cdot \hat{H} + X, \tag{4.3}$$

$$X = i \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho \cdot \hat{H}) \frac{1}{\sin(\rho \cdot q)} \hat{s}_\rho, \tag{4.4}$$

$$M = -\frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} \frac{|\rho|^2 \cos(\rho \cdot q)}{\sin^2(\rho \cdot q)} (\hat{s}_\rho - 1_\mathcal{D}) + i \sum_{\rho \in \Delta_+} g_{|\rho|} \frac{(\rho \cdot \hat{H})^2}{\sin^2(\rho \cdot q)}, \tag{4.5}$$

and *diagonal matrices*:

$$Q = q \cdot \hat{H} : \quad (Q)_{\alpha\beta} = (q \cdot \alpha) \delta_{\alpha\beta}, \tag{4.6}$$

$$R = e^{2iQ}. \tag{4.7}$$

Again L and Q are hermitian $L^\dagger = L$, $Q^\dagger = Q$ and M is anti-hermitian $M^\dagger = -M$. Thus R is unitary.

As shown in [10] (I5.3a), (I5.3b), when the root system admits a minimal representation, and $\mathcal{R}$ being the set of minimal weights, the time evolution along the flow of the Hamiltonian (4.1) displays:

$$\frac{\partial L}{\partial t} = [L, M], \tag{4.8}$$

$$\frac{\partial R}{\partial t} = [R, M] + i (RL + LR). \tag{4.9}$$

With the same definition of the unitary matrix $U(t)$ as above (2.13),

$$\frac{\partial U}{\partial t} = UM, \qquad U(0) = 1_\mathcal{D}, \tag{4.10}$$

we introduce a matrix

$$\mathcal{W}(t) = U(t)R(t)\,U(t)^{-1} = U(t)\,\mathrm{e}^{2iQ(t)}\,U(t)^{-1}. \tag{4.11}$$

It satisfies a simple first order linear differential equation

$$\frac{\partial \mathcal{W}}{\partial t} = U(\partial R/\partial t - [R, M])U^{-1} = i\,U(RL + LR)U^{-1} \tag{4.12}$$

$$= i\left(\mathcal{W}ULU^{-1} + ULU^{-1}\mathcal{W}\right), \tag{4.13}$$

since as in (2.15), (2.16), ULU^{-1} is a constant matrix:

$$\frac{\partial}{\partial t}\left(ULU^{-1}\right) = U(\partial L/\partial t - [L, M])U^{-1} = 0, \tag{4.14}$$

$$U(t)L(t)U(t)^{-1} = L(0). \tag{4.15}$$

The solution is

$$\mathcal{W}(t) = \mathrm{e}^{itL(0)}\mathrm{e}^{2iQ(0)}\mathrm{e}^{itL(0)}. \tag{4.16}$$

By *diagonalising* the above matrix solution, we obtain the explicit solution $\{q(t)\}$ of the classical Sutherland model (4.1). One might naturally wonder if the coordinates $\{q(t)\}$ could be determined uniquely from the unitary matrix (4.16). The answer is affirmative since the motion is always restricted to one of the Weyl alcoves due to the periodicity and singularity of the potential. Near the boundary of a Weyl alcove, for example at $\rho \cdot q = 0$, $\rho \in \Delta$, the singularity of the potential $\sim 1/(\rho \cdot q)^2$ can never be surpassed classically. Therefore if $\{q(0)\}$ is in the principal Weyl alcove,

$$PW_T = \{q \in \mathbf{R}^r | \rho \cdot q > 0, \quad \rho \in \Pi, \quad \rho_h \cdot q < \pi\}, \tag{4.17}$$

$\{q(t)\}$ will always remain there. Here Π is the set of the simple roots and ρ_h is the highest weight. This removes any ambiguity in determining $\{q(t)\}$ from the eigenvalues of (4.16). As in the rational potential cases, the spectrum of $\mathcal{W}(t)$ is highly constrained as a consequence of the Weyl invariance (2.20), (2.21).

4.2 Universal Lax pair

The universal Lax pair has $\cot(\rho \cdot q)$ function in L instead of $1/\sin(\rho \cdot q)$ in (4.4)

$$L = p \cdot \hat{H} + X, \tag{4.18}$$

$$X = i \sum_{\rho \in \Delta_+} g_{|\rho|}\,(\rho \cdot \hat{H})\cot(\rho \cdot q)\,\hat{s}_\rho, \tag{4.19}$$

$$M = -\frac{i}{2}\sum_{\rho \in \Delta_+} g_{|\rho|}\frac{|\rho|^2}{\sin^2(\rho \cdot q)}\,(\hat{s}_\rho - 1_D)\,, \tag{4.20}$$

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which satisfy $\partial L/\partial t = [L, M]$ for the Hamiltonian flow but the additional equation (4.9) takes a different form.

For $\mathcal{R}$ being the set of minimal weights, it reads

$$\frac{\partial R}{\partial t} = [R, M] + i\left(R(L + K) + (L - K)R\right), \tag{4.21}$$

in which K is a non-negative constant matrix commuting with M:

$$K \equiv \sum_{\rho \in \Delta_+} g_{|\rho|}(\rho \cdot \hat{H})(\rho^\vee \cdot \hat{H})\hat{s}_\rho, \quad [K, M] = 0. \tag{4.22}$$

It is a very important quantity in Calogero-Moser systems appearing in many contexts. For example, it is a commutator of Q (2.10) and the rational Lax matrix L (2.8), (2.7) (see (4.36) of [15] and (2.40) of [16]):

$$[Q, L] = iK. \tag{4.23}$$

It should be noted that if K is defined as above, the expression (4.22) is *universal*, that is valid for any root system Δ and any choice of $\mathcal{R}$. Various properties of the K matrix, whose eigenvalues are all 'integers', are discussed in detail by Corrigan-Sasaki, in the Appendix of [16].

For $\mathcal{R}$ being the set of all roots Δ (for the simply-laced root systems) or the set of short roots Δ_S (for non simply-laced root systems) and also the set of vector weights ($\mathbf{V}$) for the C, the relation corresponding to (4.9) and (4.21) reads

$$\frac{\partial R}{\partial t} = [R, M] + i\left(R(L + \tilde{K}) + (L - \tilde{K})R\right), \tag{4.24}$$

in which $\tilde{K}$ is another constant matrix commuting with M,

$$\tilde{K} = \sum_{\rho \in \Delta_+} g_{|\rho|}|\rho \cdot \hat{H}|\hat{s}_\rho, \qquad [\tilde{K}, M] = 0, \tag{4.25}$$

introduced by Corrigan-Sasaki, as (5.32) of [16].

Now the explicit solution of the Sutherland system is achieved for any crystallographic root system, since one can choose at least one Lax pair satisfying (4.24). We proceed as before by defining the unitary matrix $U(t)$ by (4.10) and introduce a matrix

$$\mathcal{W}(t) = U(t)R(t)\,U(t)^{-1} = U(t)\,\mathrm{e}^{2iQ(t)}\,U(t)^{-1}. \tag{4.26}$$

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It satisfies a simple first order linear differential equation

$$\frac{\partial \mathcal{W}}{\partial t} = U(\partial R/\partial t - [R, M])U^{-1} = i\, U(R(L + \tilde{K}) + (L - \tilde{K})R)U^{-1} \qquad (4.27)$$

$$= i\left(\mathcal{W}U(L + \tilde{K})\,U^{-1} + U(L - \tilde{K})\,U^{-1}\mathcal{W}\right), \qquad (4.28)$$

since as in (2.15), (2.16), $U(L \pm \tilde{K})\,U^{-1}$ is a constant matrix:

$$\frac{d}{dt}\left(U(L \pm \tilde{K})\,U^{-1}\right) = U(\partial L/\partial t - [L, M])U^{-1} = 0, \qquad (4.29)$$

$$U(t)(L(t) \pm \tilde{K})U(t)^{-1} = L(0) \pm \tilde{K}. \qquad (4.30)$$

The solution is

$$\mathcal{W}(t) = e^{it(L(0) - \tilde{K})}e^{2iQ(0)}e^{it(L(0) + \tilde{K})}. \qquad (4.31)$$

By *diagonalising* the above matrix solution, we obtain the explicit solution $\{q(t)\}$ of the classical Sutherland system (4.1) for any root system.

The very fact that $\tilde{K}$ (K) commutes with M simply means that a spectral parameter λ can be introduced trivially into the Lax pair for degenerate potentials [8]:

$$L_\lambda \equiv L + \lambda\tilde{K}, \quad \dot{L}_\lambda = [L_\lambda, M]. \qquad (4.32)$$

5 Rational Higher Flows

The integrable hierarchy of the Calogero-Sutherland systems consists of Hamiltonians generated by higher conserved quantities, which are constructed, for example, from the trace of the higher powers of the L matrix, $\mathcal{H}_n \propto \text{Tr}(L^{2n})$. The method of explicit integration as described in the preceding sections applies also to these higher Hamiltonian flows, as shown by Suris [7] for the A type root systems with the conventional Lax pair, $\mathcal{R} = \mathbf{V}$. However, in contrast to the basic Calogero-Sutherland flows, it works only for those systems based on the classical root systems, the A, B, C and D. Let us denote by $\mathcal{R}$ the set of vector weights $\mathbf{V}$, for the A, C and D root systems and the set of short roots Δ_S for the B root system. These particular sets $\mathcal{R}$ have unique orthogonality property

$$\text{if } \mu \neq \pm\nu, \quad \mu \cdot \nu = 0, \quad \forall \mu, \nu \in \mathcal{R}, \qquad (5.1)$$

which endows a very special structure to the Lax pair represented on $\mathcal{R}$. The dimensions of the corresponding Lax matrices are $\mathcal{D} = r + 1$ for the A_r and $\mathcal{D} = 2r$ for the B_r, C_r and D_r.

It is through these special Lax matrices that the explicit integration of the higher rational and trigonometric flows is realised.

Let us start with the explicit forms of the rational L matrices:

$$(A): \qquad L, \qquad L_{jk} = p_j \delta_{jk} + ig(1 - \delta_{jk})/(q_j - q_k), \qquad (5.2)$$

$$(B): \quad L = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}, \quad A_{jk} = p_j \delta_{jk} + ig_L(1 - \delta_{jk})/(q_j - q_k),$$

$$B_{jk} = i(g_S/q_j)\delta_{jk} + ig_L(1 - \delta_{jk})/(q_j + q_k). \qquad (5.3)$$

The rational C system will not be discussed since it is equivalent to the rational B system. The rational D system is obtained by constraining $g_S = 0$ in the rational B system.

The higher Hamiltonians are

$$(A): \qquad \mathcal{H}_n = \mathrm{Tr}(L^{n+1})/(n+1), \qquad n \geq 1, \qquad (5.4)$$

$$(B, D): \qquad \mathcal{H}_n = \mathrm{Tr}(L^{2n})/(4n), \qquad n \geq 1. \qquad (5.5)$$

The lowest $\mathcal{H}_1$ is the original Hamiltonian (2.3). The basic idea is to rewrite the Hamiltonian flow

$$\frac{\partial q_j}{\partial t_n} = \frac{\partial \mathcal{H}_n}{\partial p_j}, \qquad \frac{\partial p_j}{\partial t_n} = -\frac{\partial \mathcal{H}_n}{\partial q_j}, \qquad (5.6)$$

into equivalent matrix forms

$$\frac{\partial L}{\partial t_n} = [L, M_n], \qquad (5.7)$$

$$\frac{\partial Q}{\partial t_n} = [Q, M_n] + L^n \left(L^{2n-1}\right), \qquad (5.8)$$

as in the lowest flow (2.11), (2.12).

In contrast to the lowest flow case in which the explicit form of M is given (2.9), we can interpret part of (5.7) and (5.8) as determining M_n. The diagonal part of the Q equation (5.8) is equivalent to the first half of the canonical equations (5.6). The off-diagonal part of the Q equation (5.8) determines the off-diagonal part of M_n completely:

$$(M_n)_{\mu\nu} = -(L^n)_{\mu\nu}/q \cdot (\mu - \nu) \quad \left(-(L^{2n-1})_{\mu\nu}/q \cdot (\mu - \nu)\right), \qquad \mu \neq \nu. \qquad (5.9)$$

Whereas the diagonal part of M_n does not enter the the Q equation (5.8), it can be determined from the off-diagonal part of the Lax equation. The result is very simple:

$$(M_n)_{\mu\mu} = -\sum_{\nu \neq \mu}(M_n)_{\nu\mu} = -\sum_{\nu \neq \mu}(M_n)_{\mu\nu}, \quad M_n^\dagger = -M_n. \qquad (5.10)$$

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The proof that the diagonal part of the higher flow Lax equation (5.7) is equivalent to the second half of the canonical equations (5.6) goes almost parallel to that of the lowest flow.

After the equivalence of the canonical equations (5.6) with the two matrix equations (5.7) and (5.8) is established, the explicit integration by diagonalisation is straightforward. Let us define a $\mathcal{D} \times \mathcal{D}$ unitary matrix $U_n(t_n)$ by the linear equation and the initial condition:

$$\frac{\partial U_n}{\partial t_n} = U_n M_n, \qquad U_n(0) = 1_{\mathcal{D}}. \tag{5.11}$$

Then a matrix function $W_n(t_n)$, defined by

$$W_n(t_n) \equiv U_n(t_n) Q(t_n) U_n^{-1}(t_n), \tag{5.12}$$

has a simple time evolution

$$\frac{\partial W_n}{\partial t_n} = U_n(\partial Q/\partial t_n - [Q, M_n])U_n^{-1} = \left(U_n L U_n^{-1}\right)^n \quad \left((U_n L U_n^{-1})^{2n-1}\right), \tag{5.13}$$

$$\frac{\partial}{\partial t_n}\left(U_n L \, U_n^{-1}\right) = U_n(\partial L/\partial t_n - [L, M_n])U_n^{-1} = 0, \tag{5.14}$$

$$\Longrightarrow U_n(t_n) L(t_n) U_n(t_n)^{-1} = L(0). \tag{5.15}$$

The solution is

$$W_n(t_n) = W_n(0) + t_n \, \partial W_n(0)/\partial t_n, \tag{5.16}$$

with the initial values

$$W_n(0) = Q(0), \quad \partial W_n(0)/\partial t_n = L(0)^n \quad (L(0)^{2n-1}), \tag{5.17}$$

which are determined by the initial values of the canonical variables $q_j(0)$, $p_j(0)$, $j = 1, \ldots, r$. Due to the defining relation of $W_n(t_n)$ in terms of the *diagonal matrix* $Q(t_n)$ (5.12), the solution $\{q(t_n)\}$ of the the canonical equations of motion (5.6) with the above Hamiltonian (5.4) or (5.5), is simply obtained by *diagonalising* the above matrix solution (5.16). Determination of the conjugate momenta $\{p(t_n)\}$ requires solution of the second half of the canonical equations of motion (5.6), which are now *algebraic* since $\{\partial q/\partial t_n\}$ are now known functions of time. Extension to the generic higher flows of the hierarchy

$$\mathcal{H} = \sum_n c_n \mathcal{H}_n, \quad c_n: \; const., \tag{5.18}$$

is straightforward since the matrix equations (5.7) and (5.8) are linear in M_n. However, some higher flows cannot be treated this way. For example, in the D_r (r: odd) theory, there exists another conserved quantity (Hamiltonian) of the form $p_1 p_2 \cdots p_r + \cdots$, which cannot be written as (5.18).

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6 Trigonometric Higher Flows

The basic logics of the explicit integration of the trigonometric higher flows is almost the same as that of the rational higher flows, except that we have to consider two different types of Lax pairs; the minimal and the universal. So we just write down the key formulas without detailed derivation.

6.1 Minimal Lax Pair

We discuss the explicit integration of the trigonometric higher flows of the A and D theory in terms of the minimal Lax pair, although the formulation in terms of the universal Lax pair works well for them, too.

The explicit forms of the trigonometric minimal L matrices are:

$$(A): \qquad L, \qquad L_{jk} = p_j \delta_{jk} + ig(1 - \delta_{jk})/\sin(q_j - q_k), \qquad (6.1)$$

$$(D): \qquad L = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}, \qquad A_{jk} = p_j \delta_{jk} + ig(1 - \delta_{jk})/\sin(q_j - q_k),$$

$$B_{jk} = ig(1 - \delta_{jk})/\sin(q_j + q_k). \qquad (6.2)$$

The higher Hamiltonians take exactly the same form as (5.4) and (5.5). The lowest $\mathcal{H}_1$ is the original Hamiltonian (4.1). We rewrite the higher Hamiltonian flow (5.6) into equivalent matrix forms

$$\frac{\partial L}{\partial t_n} = [L, M_n], \qquad (6.3)$$

$$\frac{\partial R}{\partial t_n} = [R, M_n] + i(RL^n + L^n R) \quad \left(i(RL^{2n-1} + L^{2n-1}R)\right), \qquad (6.4)$$

as in the lowest flow (4.8), (4.9). The off-diagonal part of M_n is

$$(M_n)_{\mu\nu} = -(L^n)_{\mu\nu} \cot[q \cdot (\mu - \nu)] \quad \left(-(L^{2n-1})_{\mu\nu} \cot[q \cdot (\mu - \nu)]\right), \qquad \mu \neq \nu. \qquad (6.5)$$

The diagonal part is

$$(M_n)_{\mu\mu} = \sum_{\nu \neq \mu} (L^n)_{\nu\mu}/\sin[q \cdot (\nu - \mu)] = \sum_{\nu \neq \mu} (L^n)_{\mu\nu}/\sin[q \cdot (\mu - \nu)], \quad M_n^\dagger = -M_n. \qquad (6.6)$$

The $\mathcal{D} \times \mathcal{D}$ matrix $\mathcal{W}_n(t_n)$ obeys simple time evolution:

$$\mathcal{W}_n(t_n) = U_n(t) R(t_n) U_n(t_n)^{-1} = U_n(t_n) e^{2iQ(t_n)} U_n(t_n)^{-1}, \qquad (6.7)$$

$$= e^{it_n L(0)^n} e^{2iQ(0)} e^{it_n L(0)^n} \quad \left(e^{it_n L(0)^{2n-1}} e^{2iQ(0)} e^{it_n L(0)^{2n-1}}\right). \qquad (6.8)$$

By *diagonalising* the above matrix solution (6.8), we obtain the explicit solution $\{q(t_n)\}$ of the higher flows of the Sutherland system (5.4) and (5.5) for the A and D root systems.

6.2 Universal Lax Pair

The explicit integration of the higher flows of the B and C Sutherland systems is achieved in terms of the universal Lax pairs based on the set of short roots ($\mathcal{R} = \Delta_S$) for B and the set of vector weights ($\mathcal{R} = \mathbf{V}$) for C. For the rank r system both have $\mathcal{D} = 2r$.

The Lax matrix L and the constant matrix $\tilde{K}$ (4.25) are:

$$L = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}, \qquad \tilde{K} = \begin{pmatrix} S & T \\ T & S \end{pmatrix}, \tag{6.9}$$

$$(B): A_{jk} = p_j \delta_{jk} + ig_L(1 - \delta_{jk})\cot(q_j - q_k), \qquad S_{jk} = g_L(1 - \delta_{jk}), \tag{6.10}$$

$$B_{jk} = ig_S \cot q_j\, \delta_{jk} + ig_L(1 - \delta_{jk})\cot(q_j + q_k), \quad T_{jk} = g_S\delta_{jk} + g_L(1 - \delta_{jk}), \tag{6.11}$$

$$(C): A_{jk} = p_j\delta_{jk} + ig_S(1 - \delta_{jk})\cot(q_j - q_k), \qquad S_{jk} = g_S(1 - \delta_{jk}), \tag{6.12}$$

$$B_{jk} = 2ig_L \cot 2q_j\, \delta_{jk} + ig_S(1 - \delta_{jk})\cot(q_j + q_k), \quad T_{jk} = 2g_L\delta_{jk} + g_S(1 - \delta_{jk}). \tag{6.13}$$

It is easy to see

$$e^{iQ}(L + \tilde{K})e^{-iQ} = e^{-iQ}(L - \tilde{K})e^{iQ}, \tag{6.14}$$

$$\Rightarrow \operatorname{Tr}(L + \tilde{K})^n = \operatorname{Tr}(L - \tilde{K})^n, \tag{6.15}$$

which are conserved quantities of the Sutherland flow (4.1). It differs from the usual one $\operatorname{Tr}(L^n)$ by a linear combination of lower order conserved quantities. The canonical equations of the higher flow Hamiltonian

$$\mathcal{H}_n = \operatorname{Tr}\left((L \pm \tilde{K})^{2n}\right)/(4n) \tag{6.16}$$

are equivalent to the matrix equations

$$\frac{\partial L}{\partial t_n} = [L, M_n], \tag{6.17}$$

$$\frac{\partial R}{\partial t_n} = [R, M_n] + i\left(R(L + \tilde{K})^{2n-1} + (L - \tilde{K})^{2n-1}R\right). \tag{6.18}$$

The off-diagonal part of M_n is

$$(M_n)_{\mu\nu} = -\left[e^{iq\cdot(\mu-\nu)}(L + \tilde{K})^n{}_{\mu\nu} + e^{-iq\cdot(\mu-\nu)}(L - \tilde{K})^n{}_{\mu\nu}\right]/\sin[q\cdot(\mu - \nu)], \quad \mu \neq \nu. \tag{6.19}$$

The diagonal part is

$$(M_n)_{\mu\mu} = -\sum_{\nu\neq\mu}(M_n)_{\nu\mu} = -\sum_{\nu\neq\mu}(M_n)_{\mu\nu}, \quad M_n^\dagger = -M_n. \tag{6.20}$$

The $\mathcal{D}\times\mathcal{D}$ matrix $\mathcal{W}_n(t_n)$ obeys simple time evolution:

$$\begin{aligned}
\mathcal{W}_n(t_n) &= U_n(t)R(t_n)\,U_n(t_n)^{-1} = U_n(t_n)\,\mathrm{e}^{2iQ(t_n)}\,U_n(t_n)^{-1}, \tag{6.21}\\
&= \mathrm{e}^{it_n(L(0)-\tilde{K})^{2n-1}}\mathrm{e}^{2iQ(0)}\mathrm{e}^{it_n(L(0)+\tilde{K})^{2n-1}}. \tag{6.22}
\end{aligned}$$

By *diagonalising* the above matrix solution (6.22), we obtain the explicit solution $\{q(t_n)\}$ of the higher flows of the Sutherland system (6.16) for the B and C root systems.

7　Summary and Comments

Explicit integration of the Calogero and Sutherland systems by means of diagonalisation is demonstrated for any root system, the exceptional as well as the classical and the non-crystallographic. It is based on the universal Lax pair for the degenerate potentials, that is the rational with/without the harmonic confining potential and the trigonometric/hyperbolic potentials. As emphasised in the text, it is very closely related to the concept of algebraic linearisation by Caseiro-Françoise-Sasaki [10]. The method is extended to the higher Hamiltonian flows of the rational and trigonometric/hyperbolic interactions. In contrast to the basic Calogero-Sutherland flows, the applicability is limited to those systems based on the classical root systems, the A, B, C and D root systems.

The theory of explicit integration of higher Hamiltonian flows is very closely related to the dynamical r-matrix [7, 17, 18] and the Hamiltonian reduction [19, 13]. In the case of the most classical rational potential of the A type, the method of Hamiltonian reduction starts from the large phase space of the matrix dynamical variable W and its conjugate momentum variable Z, which are both assumed to be Hermitian. The Hamiltonians

$$\mathcal{H}_n = \mathrm{Tr}(Z^{n+1})/(n+1) \tag{7.1}$$

generate the flows

$$\frac{\partial W}{\partial t_n} = Z^n, \qquad \frac{\partial Z}{\partial t_n} = 0. \tag{7.2}$$

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This Hamiltonian system in invariant under the action $(W, Z) \to (UWU^{-1}, UZU^{-1})$ of unitary matrices U. The reduced phase space is obtained by imposing the constraint

$$[W, Z] = iK \tag{7.3}$$

and factoring out the constrained phase space by residual symmetries (i.e., by the group of unitary matrices that commute with K). The (Q, L) pair (4.23) is nothing but a representative of a point of the reduced phase space, which is connected with the point (W, Z) of the large phase space by a (t-dependent) unitary matrix U as

$$Q = U^{-1}WU, \qquad L = U^{-1}ZU. \tag{7.4}$$

The linear flows of (W, Z) are thereby mapped to the Calogero flows of (Q, L). This is the way to understand the rational Calogero system of the A type as a Hamiltonian reduction [19]; a similar interpretation has been proposed for a few other cases [13]. The dynamical r-matrix has been constructed in this framework of Hamiltonian reduction [7]. We expect that all the cases discussed in this paper can be treated in the same way.

Acknowledgements

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